Problem 1: averaging theorems (50 points).

We showed in class that volume averages can also be computed from information on the boundary via

\[
\langle \epsilon \rangle = \frac{1}{V} \int_{\partial \Omega} \text{sym}(u \otimes n) \, dS \approx \frac{1}{V} \sum_{a=1}^{n} \text{sym}(u^a \otimes \tilde{n}^a) \quad \text{with} \quad \tilde{n}^a = \frac{1}{2} \sum_{e} n_e^a A_e
\]

and

\[
\langle \sigma \rangle = \frac{1}{V} \int_{\partial \Omega} t \otimes x \, dS \approx \frac{1}{V} \sum_{a} F^a \otimes x^a,
\]

where summation over \( a \) indicates the sum over all boundary nodes (see lecture notes). \( n^a_e \) is the element outward normal vector at node \( a \) seen from element \( e \) (with surface area or length \( A_e \)). \( F^a \) is the total force at node \( a \). We use linearized kinematics for convenience.

Let us verify those relations computationally, using our finite element code.

To this end, construct a square-shaped 2D RVE (use, e.g., CST elements) and a linear elastic material model in 2D. Create two different material models which differ in their moduli. Next, create a composite by randomly assigning one of two materials to 50% of the elements in your RVE. Apply affine displacement BCs

\[
u(x) = \varepsilon_0 x \quad \text{for} \quad x \in \partial \Omega
\]

to every node on the outer boundary of the RVE for some deformation gradient \( \varepsilon_0 \neq 0 \). Solve the boundary value problem (using, e.g., the Newton-Raphson solver).

Once you have the solution, let us verify the two above averaging theorems as follows:

1. Use methods computeStressesAtGaussPoints and computeStrainsAtGaussPoints (these exist inside the isoparametric element) to compute the averages \( \langle \epsilon \rangle \) and \( \langle \sigma \rangle \) by summing over all elements in your FE mesh. Verify that indeed \( \langle \epsilon \rangle = \varepsilon_0 \).
2. Compute \( \langle \epsilon \rangle \) by using the above formula, summing over all boundary nodes.
3. Compute \( \langle \sigma \rangle \) by using the above formula, summing over all boundary nodes.

Hint: Instead of computing boundary normals in a general fashion, simply exploit the fact that your RVE is a square, so that you can sort boundary nodes by one of the four boundaries, on each of which the normal is known. Also, all element surface areas/lengths and all element volumes are known and the same in a regular mesh.
Problem 2: representative volume elements (50 points).

Let us use the same setup as above and try to derive the effective stiffness tensor of the composite. The stiffness tensor $\hat{C}^*$ in Voigt notation can be obtained from the linear system of equations

$$\langle \tilde{\sigma} \rangle = \hat{C}^* \tilde{\varepsilon}_0.$$ 

Note that in 2D these are only 3 equations for the 9 unknown components of $\hat{C}^*$ (one cannot a-priori assume symmetry of $C^*$). Therefore, pick three different (linearly independent) vectors $\tilde{\varepsilon}_0 \neq 0$ (since the system is linear, the choice of $\varepsilon_0$ is irrelevant) and for each choice compute $\langle \tilde{\sigma} \rangle$. Use Eigen or Matlab or Mathematica to compute the components of $\hat{C}^*$ from the three computed $\langle \tilde{\sigma} \rangle$-vectors and the known corresponding three $\tilde{\varepsilon}_0$-vectors by solving the above linear system of equations.

Let us perform the following case study:

(i) Use the above setup to compute $\hat{C}^*$ for a fixed number of elements $n_e$ but by averaging over an increasing number $N$ of random RVE realizations (i.e., simply re-run the analysis with a new random seed for the material model assignments). By plotting the mean and standard deviation of, e.g., $\hat{C}_{11}$ vs. $n_e$ show the convergence of $\hat{C}^*$ with ensemble enlargement.

(ii) Repeat the above with finer meshes to show the effect to sample enlargement.

Hint: Note that you can automatize the re-running of the above analysis by having a for-loop around your entire Main routine and outputting the average stresses or moduli into files with reasonably-changing file names. No need to run each realization or simulation by hand.

total: 100 points